TAYLOR IS PRIME

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ABSTRACT. We study the Taylor varieties and obtain new characterizations of them via compatible reflexive digraphs. Based on our findings, we prove that in the lattice of interpretability types of varieties, the filter of the types of all Taylor varieties is prime.

1. INTRODUCTION

We require some basic concepts to introduce the topic of our investigations in this paper. Let A be a set. An n-ary operation f on A is *idempotent* if $f(a, \ldots, a) = a$ for all $a \in A$. A *clone on A* is a set of operations on A that contains all projection operations and is closed under composition. Let $\mathbf{A}$ be an algebra. The *clone of $\mathbf{A}$* denoted by $\mathrm{Clo}(\mathbf{A})$ is the clone of term operations of $\mathbf{A}$. The *clone of a variety $\mathcal{V}$* is the clone of the free algebra generated by countably many free generators. An *algebra (a variety) is idempotent* if all operations in its clone are idempotent.

A *clone homomorphism* from a clone C to a clone D is a map from C to D that maps each projection of C to the corresponding projection of D and commutes with composition. A *variety $\mathcal{V}$ interprets in a variety $\mathcal{W}$* if there is a clone homomorphism from the clone of $\mathcal{V}$ to the clone of $\mathcal{W}$.

As easily seen, interpretability is a quasiorder on the class of varieties. The blocks of this quasiorder are called *interpretability types*. In [1] Garcia and Taylor introduced the *lattice $\mathcal{L}$ of interpretability types of varieties* that is obtained by taking the quotient of the class of varieties quasiordered by interpretability and the related equivalence. The join in $\mathcal{L}$ is described as follows. Let $\mathcal{V}_1$ and $\mathcal{V}_2$ be two varieties of disjoint signatures. Let $\mathcal{V}_1$ and $\mathcal{V}_2$ be defined by the sets Σ_1 and Σ_2 of identities, respectively. *Their join $\mathcal{V}_1 \vee \mathcal{V}_2$ is the variety defined by $\Sigma_1 \cup \Sigma_2$.* The so defined join is compatible with the interpretability relation of varieties, and naturally yields the definition of the join operation in $\mathcal{L}$.

A *strong Maltsev condition* is a finite set of identities in the language of a variety. A *Maltsev condition* is a sequence M_n, $n \geq 1$, of strong Maltsev conditions where the variety defined by M_{n+1} interprets in the variety defined by M_n for each $n \geq 1$. We say that a *variety $\mathcal{V}$ admits a Maltsev condition M_n, $n \geq 1$, (or the terms that occur in a Maltsev condition M_n, $n \geq 1$)* if there exists an n such that the variety defined by M_n interprets in $\mathcal{V}$. By a *filter*, we mean an upwardly closed sublattice of a lattice. Clearly, the types of the varieties that admit a particular Maltsev condition form a filter in $\mathcal{L}$. Such filters of $\mathcal{L}$ are called *Maltsev filters*. A filter F of a lattice is called *prime* if F is a proper subset of the lattice and for any

two elements $a, b \notin F$, we have $a \vee b \notin F$. To gain information on the structural properties of $\mathcal{L}$, it is useful to determine which of the well-known classical Maltsev filters are prime filters in $\mathcal{L}$. In case a Maltsev filter is (not) prime, we just say sometimes that the related Maltsev condition is (not) prime.

In [1], primeness of some well-known Maltsev filters of $\mathcal{L}$ was decided. For example, in [1], it was proved that the filter containing the largest element of $\mathcal{L}$, given by the Maltsev condition $\{x = y\}$, the filter determined by the Maltsev condition $\{s(x, y) = s(y, x)\}$, and the filter determined by congruence regularity are all prime in $\mathcal{L}$. However, the filter determined by congruence distributivity is not prime in $\mathcal{L}$. The primeness of some Maltsev filters, such as the filter of the types of congruence permutable varieties and the filter of the types of congruence modular varieties, were stated as open problems in [1]. Not long ago, the last three authors of this paper proved that congruence permutability is indeed a prime Maltsev condition, see [3]. The problem on congruence modular varieties is still open.

By extending the results of Hobby and McKenzie in [4], Kearnes and Kiss elaborated a hierarchy of varieties based on certain Maltsev conditions in [5]. It is natural to look at the problem of primeness for the Maltsev filters in $\mathcal{L}$ that were characterized in [5]. The largest Maltsev filter described in [5], introduced by Taylor in [11], and studied in [4] is the class of the interpretability types of Taylor varieties (*Taylor interpretability types* for short). In the present paper, we investigate the class of Taylor varieties, and prove that the filter of Taylor interpretability types is prime in $\mathcal{L}$.

In Figure 1, we depicted the Maltsev filters of $\mathcal{L}$ that were used to establish a certain hierarchy of varieties in [5]. A node is painted black if it symbolizes a prime Maltsev filter, is left empty if it symbolizes a non-prime filter, and is painted gray if its status is unknown at present. The filters in the figure are depicted accordingly to reverse containment. This may be unusual but we want to keep the direction coming from the interpretability ordering. It is well known that the filters in the figure are pairwise different.
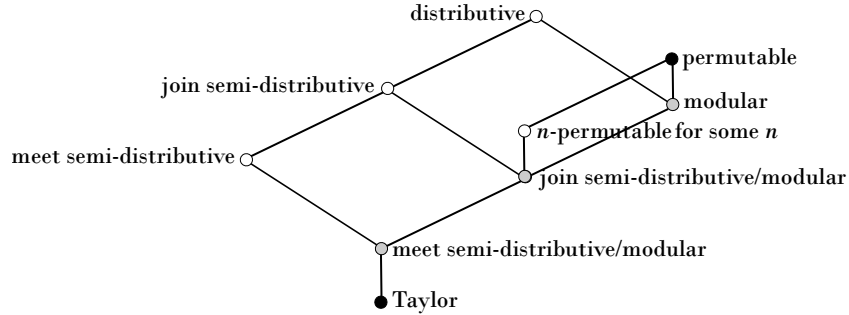

FIGURE 1. Maltsev filters that establish a hierarchy of varieties.

In a short while below, we define Taylor varieties. Then the meaning of the filter of Taylor interpretability types that is the main object of our investigations in the present paper becomes clear. For the definitions of the other Maltsev filters in Figure 1, the reader should consult [5]. Note that monograph [5] contains a wide variety of characterizations of the filter of Taylor interpretability types and most of the other filters in the figure.

We remark that the non-primeness of n-permutability for some n was proved by the last three authors of the present paper in [2]. To verify the status of distributivity, join and meet semi-distributivity in Figure 1, we render a proposition.

We need the following definitions. A lattice is *meet semi-distributive* if it satisfies the implication $x \wedge y = x \wedge z \Rightarrow x \wedge y = x \wedge (y \vee z)$. The *join semi-distributivity* of a lattice is defined dually. A variety $\mathcal{V}$ is *congruence meet semi-distributive (congruence join semi-distributive, congruence distributive)* if the congruence lattice of any algebra in $\mathcal{V}$ has the same property. A ternary term t in the language of a variety $\mathcal{V}$ is called a *a majority term for $\mathcal{V}$* if t satisfies the identities $t(y, x, x) = t(x, y, x) = t(x, x, y) = x$ in $\mathcal{V}$.

Proposition 1.1. *The following Maltsev conditions are not prime:*

  (1) congruence meet semi-distributivity,
  (2) congruence join semi-distributivity,
  (3) congruence distributivity,
  (4) admittance of a majority term.

Proof. As well known, see for example [5], any variety that satisfies a Maltsev condition on the above list also satisfies the ones prior to it. So to prove the proposition, it suffices to present two varieties that are not meet semi-distributive such that their join admits a majority term.

Let $\mathcal{V}_1$ be the variety defined by the identities

$$m(x, y, y) = m(y, x, y) = m(y, y, x) = x$$

for a single ternary operation symbol m, and $\mathcal{V}_2$ the variety defined by the identities

$$s(x, x) = x \text{ and } s(x, y) = s(y, x)$$

for a single binary operation symbol s.

We take the algebras

$$\mathbf{A}_1 = (Z_2; \ x + y + z) \in \mathcal{V}_1 \text{ and } \mathbf{A}_2 = (Z_3; \ 2x + 2y) \in \mathcal{V}_2$$

and determine the congruence lattices of the algebras $\mathbf{A}_i^2 \in \mathcal{V}_i$, $1 \leq i \leq 2$. Notice that these squares have the same polynomials as the groups $\mathbf{Z}_{i+1}^2 = (Z_{i+1}; \ +)^2$, $1 \leq i \leq 2$, respectively. So $\mathrm{Con}(\mathbf{A}_i^2) \cong \mathrm{Con}(\mathbf{Z}_{i+1}^2) \cong \mathbf{M}_{i+2}$, $1 \leq i \leq 2$, where $\mathbf{M}_j$ denotes the $(j+2)$-element lattice with j atoms. Since these congruence lattices are not meet semi-distributive, the varieties $\mathcal{V}_1$ and $\mathcal{V}_2$ are not meet semi-distributive either, but their join has a majority term of the form $m(s(x, y), s(x, z), s(y, z))$. $\square$

We remark that the question of primeness of an idempotent Maltsev condition makes sense restricted to the sublattice $\mathcal{L}_{\mathrm{Id}}$ of interpretability types of idempotent varieties in $\mathcal{L}$. While the primeness of an idempotent Maltsev condition in $\mathcal{L}$ implies the primeness of the same idempotent Maltsev condition in $\mathcal{L}_{\mathrm{Id}}$, the converse does not hold in general. In this respect, we mention that in the proof of the above proposition, only idempotent varieties were used, so the proposition holds if we consider interpretability in $\mathcal{L}_{\mathrm{Id}}$, that is, majority, distributivity, join semi-distributivity, and meet semi-distributivity are not prime Maltsev conditions in $\mathcal{L}_{\mathrm{Id}}$ as well. The primeness of the filter of Taylor interpretability types in $\mathcal{L}_{\mathrm{Id}}$ follows from Taylor's work in [11]. We shall give an argument below. The primeness of join semi-distributive over modular in $\mathcal{L}_{\mathrm{Id}}$ immediately follows from Lemma 9.5 in [4]. It was proved in [12] that n-permutability for some n is prime in $\mathcal{L}_{\mathrm{Id}}$. The

primeness of modularity relative to $\mathcal{L}_{\mathrm{Id}}$ was obtained in [10]. The primeness of permutability in $\mathcal{L}_{\mathrm{Id}}$ was first proved in [6]. We are not aware of any result related to the primeness of meet-semidistributive over modular in $\mathcal{L}_{\mathrm{Id}}$ or in $\mathcal{L}$.

Let $\mathcal{SET}$ denote the variety of algebras with no basic operations. We note that $\mathcal{SET}$ interprets in every variety, and hence its interpretability type is the smallest element of $\mathcal{L}$. A *Maltsev condition is trivial* if the variety of $\mathcal{SET}$ admits it. A *Taylor variety* is a variety which admits a non-trivial idempotent (strong) Maltsev condition. Taylor varieties play a crucial role in the theory of general algebras [5], in tame congruence theory [4], and in the theory of constraint satisfaction problems [7]. In all of these areas, the Taylor varieties are considered the well-behaved varieties from many points of view. Meanwhile, the non-Taylor varieties are looked on as the black sheep of the family with wild properties to be expected for them.

An identity is *linear* if it has at most one occurrence of a function symbol on each side of the identity. A Taylor term is an n-ary idempotent term t for some n such that for each $1 \leq i \leq n$, t satisfies a linear identity

$$t(\ldots, x, \ldots) = t(\ldots, y, \ldots)$$

in two different variables x and y, where the variables x and y displayed are in the i-th positions respectively on both sides, and the occurrences of the variables not displayed are arbitrarily set x or y.

The *full idempotent reduct of a variety* $\mathcal{V}$ is a variety whose signature is the set of idempotent terms of $\mathcal{V}$, and whose identities are those satisfied by the idempotent terms for $\mathcal{V}$. A *group* $\mathbf{G}$ *is compatible in a variety* $\mathcal{V}$ if there is an algebra $\mathbf{A}$ in $\mathcal{V}$ such that the underlying sets of $\mathbf{A}$ and $\mathbf{G}$ are the same and the operations of $\mathbf{A}$ commute with the group operation of $\mathbf{G}$. The following characterization of Taylor vaieties is due to Taylor, cf. Theorems 5.1, 5.2, and 5.3 in [11].

Theorem 1.2. *Let $\mathcal{V}$ be a variety, and $\mathcal{V}_{\mathrm{Id}}$ the full idempotent reduct of $\mathcal{V}$. Then the following are equivalent.*

  (1) $\mathcal{V}$ is a Taylor variety.
  (2) Every compatible group is commutative in $\mathcal{V}_{\mathrm{Id}}$.
  (3) $\mathcal{V}_{\mathrm{Id}}$ does not interpret in $\mathcal{SET}$.
  (4) $\mathcal{V}$ admits a Taylor term.

In Theorem 1.1 of [9], Olšák gave a characterization of Taylor varieties by some non-trivial idempotent strong Maltsev condition, and hence their interpretability types form a principal filter in $\mathcal{L}$. An *Olšák term* is a 6-ary idempotent term t that satisfies the following identities

$$t(x, y, y, y, x, x) = t(y, x, y, x, y, x) = t(y, y, x, x, x, y).$$

Theorem 1.3. *A variety $\mathcal{V}$ is a Taylor variety if and only if $\mathcal{V}$ admits an Olšák term.*

Further equivalent conditions characterizing Taylor varieties can be found in the monograph [5] of Kearnes and Kiss. Notice that by Theorem 1.2, if $\mathcal{V}_1$ and $\mathcal{V}_2$ are idempotent non-Taylor varieties, then they both interpret in $\mathcal{SET}$. So their join also interprets in $\mathcal{SET}$. Hence, the filter of Taylor interpretability types is prime in $\mathcal{L}_{\mathrm{Id}}$.

The main result of this paper states that we can drop idempotency from this statement, that is, the filter of Taylor interpretability types is prime in $\mathcal{L}$. For

the proof, we shall derive a novel characterization of Taylor varieties in Section 4. We present some preliminary results on disjoint unions of powers of the reflexive directed triangle digraph in Sections 2 and 3. We note at this point that the second part of Section 2 following Corollary 2.2, and the entire Section 3 cover materials that are not used in the proof of the main theorem. These parts of the paper contain motivating examples and results that led to our characterization of Taylor varieties in Section 4. So the reader may choose to skip proofs in these parts of the paper when first reading it without losing essentials to the content of Section 4.

2. Some basic facts on powers of $\mathbb{C}$

Throughout this paper, we use blackboard bold capital letters to denote relational structures, in particular digraphs, and use bold capital letters to denote algebras. Usually, we use the same capital italics to denote the base sets of these structures. We denote by $\mathbb{C}$ the 3-cycle graph together with all loops. We use the following notational convention: the vertices of $\mathbb{C}$ are labeled $0, 1, 2$ and the edges of $\mathbb{C}$ are $(0,1), (1,2), (2,0), (0,0), (1,1), (2,2)$, see Figure 2.
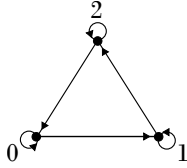

FIGURE 2. The digraph $\mathbb{C}$.

Let $\mathbb{G}$ and $\mathbb{H}$ be digraphs. An edge preserving map $\varphi \colon G \to H$ is called a *homomorphism from $\mathbb{G}$ to $\mathbb{H}$*. A homomorphism $\varphi \colon \mathbb{G} \to \mathbb{H}$ is a *retraction* if there exists a homomorphism $\varepsilon \colon \mathbb{H} \to \mathbb{G}$ where $\varphi \circ \varepsilon = \mathrm{id}_H$. A digraph $\mathbb{H}$ is a *retract* of $\mathbb{G}$ if there is a retraction from $\mathbb{G}$ onto $\mathbb{H}$. We say that $\mathbb{H}$ is a *subdigraph* of $\mathbb{G}$ if every vertex of $\mathbb{H}$ is a vertex of $\mathbb{G}$ and every edge of $\mathbb{H}$ is an edge of $\mathbb{G}$. We use $\mathbb{H} \subseteq \mathbb{G}$ to denote that $\mathbb{H}$ is a subdigraph of $\mathbb{G}$. We call $\mathbb{H}$ *a spanned subdigraph of* $\mathbb{G}$, if $\mathbb{H} \subseteq \mathbb{G}$ and every edge of $\mathbb{G}$ whose end vertices are in H is also an edge of $\mathbb{H}$. A *digraph is connected* if the reflexive, symmetric, transitive closure of its edge relation is the full relation. The *(connected) components* of a digraph are the maximal connected spanned subdigraphs.

Let $\mathbb{H} = (H, \to)$ be a digraph and I an arbitrary set. Then we write $\mathbb{H}^I$ for the digraph whose vertices are the elements of H^I, i.e., all functions from I to H, and whose set of edges is $\{(u, v) \in (H^I)^2 : \forall i \in I, \ u(i) \to v(i)\}$. One can think of I as a set of coordinates, and then $u(i)$ denotes the i-th entry of some vertex $u \in H^I$. For $J \subseteq I$ we write π_J for the projection map from H^I to H^J, i.e., $\pi_J(a) = a|_J$. This map is clearly a surjective homomorphism from $\mathbb{H}^I$ to $\mathbb{H}^J$. A homomorphism from $\mathbb{H}^I$ to $\mathbb{H}$ is called an I-ary *polymorphism* of $\mathbb{H}$.

We conceive of C^I as the Abelian group $\mathbf{Z}_3^I$. So addition and subtraction for the elements of C^I are always understood as in $\mathbf{Z}_3^I$. For some subset $J \subseteq I$ we define $\chi_J \in C^I$ by

$$\chi_J(j) = 1 \text{ if } j \in J, \text{ and } \chi_J(j) = 0 \text{ otherwise.}$$

Clearly, every $a \in C^I$ can uniquely be written as $a = \chi_A - \chi_B$ for some disjoint subsets A and B of I.

We call a digraph $\mathbb{H}$ *reflexive* if for any $u \in H$, $u \to u$. A digraph $\mathbb{H}$ is *antisymmetric* if for any $u, v \in H$, $u \to v \to u$ implies $u = v$.

Lemma 2.1. *Let $\mathbb{H}$ be an antisymmetric digraph such that any non-loop edge of $\mathbb{H}$ is contained in at most one copy of $\mathbb{C}$ in $\mathbb{H}$. Let I be a finite set, and $f \colon \mathbb{C}^I \to \mathbb{H}$ a homomorphism. Then there exist $J \subseteq I$ and an isomorphism $\iota \colon \mathbb{C}^J \to f(\mathbb{C}^I)$ such that $f = \iota \circ \pi_J$. In particular, the image of f is isomorphic to a finite power of $\mathbb{C}$.*

Proof. Let

$$J := \{j \in I \colon f \text{ depends on its } j\text{-th coordinate}\}.$$

We assume that J is non-empty, for otherwise f is a constant map and the claim obviously holds. Clearly, then there exists an onto homomorphism $\iota \colon \mathbb{C}^J \to f(\mathbb{C}^I)$ such that $f = \iota \circ \pi_J$. It suffices to prove that ι is injective and its inverse is also a homomorphism.

First we prove that ι is injective. Let us suppose to the contrary that a and b are different elements in C^J such that $\iota(a) = \iota(b)$. Then there are some $A, B \subseteq J$ such that $A \cap B = \emptyset$ and $b = a + \chi_A - \chi_B$. Without loss of generality, we may assume that A is non-empty and $j \in A$. Then $b \to a - \chi_A \to a$ in $\mathbb{C}^J$, whence $\iota(b) \to \iota(a - \chi_A) \to \iota(a)$ in $\mathbb{H}$. So by $\iota(a) = \iota(b)$ and antisymmetry in $\mathbb{H}$, $\iota(a - \chi_A) = \iota(a)$.

Then $a - \chi_A \to a - \chi_{\{j\}} \to a$ in $\mathbb{C}^J$ and $\iota(a - \chi_A) \to \iota(a - \chi_{\{j\}}) \to \iota(a)$ in $\mathbb{H}$. Hence by $\iota(a - \chi_A) = \iota(a)$ and antisymmetry, $\iota(a - \chi_{\{j\}}) = \iota(a)$. Now $a \to a + \chi_{\{j\}} \to a - \chi_{\{j\}}$ in $\mathbb{C}^J$ and $\iota(a) \to \iota(a + \chi_{\{j\}}) \to \iota(a - \chi_{\{j\}})$ in $\mathbb{H}$. By using antisymmetry again, $\iota(a + \chi_{\{j\}}) = \iota(a)$. Thus for a we have that $\iota(a - \chi_{\{j\}}) = \iota(a) = \iota(a + \chi_{\{j\}})$.

Clearly, ι depends on its j-th coordinate since f does. So there exists $c \in C^J$ such that $\iota(c - \chi_{\{j\}})$ and $\iota(c)$ are different. By antisymmetry, $\iota(c - \chi_{\{j\}})$, $\iota(c)$ and $\iota(c + \chi_{\{j\}})$ are pairwise different. So we may assume that $a(j) = c(j)$ for such c. Now there is a sequence in C^J that starts with a and ends with c such that j-th components of the members are all equal to $a(j)$ and any two consecutive members differ in a single component. This sequence clearly has two consecutive members a' and c' such that $\iota(a' - \chi_{\{j\}}) = \iota(a') = \iota(a' + \chi_{\{j\}})$, and $\iota(c' - \chi_{\{j\}})$, $\iota(c')$ and $\iota(c' + \chi_{\{j\}})$ are pairwise different. So we may assume that there is $k \neq j$ such that $a(i) = c(i)$ for all $i \in J \setminus \{k\}$, $\iota(a - \chi_{\{j\}}) = \iota(a) = \iota(a + \chi_{\{j\}})$, and $\iota(c - \chi_{\{j\}})$, $\iota(c)$ and $\iota(c + \chi_{\{j\}})$ are pairwise different. We may also assume that $a(k) = 0$ and $c(k) = 1$.

Let $\mathbb{D}$ be the subdigraph spanned by all the tuples d in $\mathbb{C}^J$ such that $d(i) = a(i)$ for all $i \in J \setminus \{j, k\}$. Let $\varphi = \pi_{\{k,j\}}|_D$. Obviously, φ is an isomorphism from $\mathbb{D}$ to $\mathbb{C}^2$. Let $\psi = \iota|_D \varphi^{-1}$. Then ψ is a homomorphism from $\mathbb{C}^2$ to $\iota(\mathbb{C}^J)$ such that by the properties of a and c, we have that $\psi(0,0) = \psi(0,1) = \psi(0,2)$, and $\psi(1,0)$, $\psi(1,1)$, $\psi(1,2)$ are pairwise different. Then

$$\psi(2,0) \to \psi(0,0) = \psi(0,1) \to \psi(1,0) \to \psi(2,0), \text{ and}$$
$$\psi(2,0) \to \psi(0,0) = \psi(0,1) \to \psi(1,2) \to \psi(2,0) \text{ in } \mathbb{H}.$$

Observe that the set S of the ψ-values appearing on the preceding two lines has less than four elements. Indeed, for otherwise the vertices appearing on each line would yield a copy of $\mathbb{C}$ in $\mathbb{H}$, and these two different copies of $\mathbb{C}$ would share the common edge $\psi(2,0) \to \psi(0,0)$, which is impossible. Then by antisymmetry of $\mathbb{H}$, S is a singleton or a 3-element set. So in both cases, $\psi(1,0) = \psi(1,2)$, a contradiction. Thus ι has to be injective.

We prove that the inverse of ι is a homomorphism. Now let a and b be arbitrary elements of C^J. Let us assume that $\iota(a) \to \iota(b)$. Put $b = a + \chi_A - \chi_B$ for some $A, B \subseteq J$, $A \cap B = \emptyset$. Then let $c = a - \chi_A - \chi_B$, and $d = a - \chi_A$. Clearly, $b \to c \to a$ and $b \to d \to a$. Therefore, we have $\iota(a) \to \iota(b) \to \iota(c) \to \iota(a)$ and $\iota(a) \to \iota(b) \to \iota(d) \to \iota(a)$. As $\mathbb{H}$ is antisymmetric and in $\mathbb{H}$ every non-loop edge is in at most one copy of $\mathbb{C}$, it follows that $\iota(c) = \iota(d)$, and thus $c = d$. This means exactly that $B = \emptyset$, and thus $a \to b$. $\square$

We note that later in the paper, typical applications of Lemma 2.1 occur when $\mathbb{H}$ is a subdigraph of some power of $\mathbb{C}$. For any relational structure $\mathbb{Q}$, let $\mathrm{Pol}(\mathbb{Q})$ denote the clone of finitary polymorphisms of $\mathbb{Q}$. By the help of the lemma, $\mathrm{Pol}(\mathbb{C})$ is easily described.

Corollary 2.2. *The clone* $\mathrm{Pol}(\mathbb{C})$ *consists of all constant operations of C and the essentially automorphism operations of* $\mathbb{C}$.

The following example shows that Lemma 2.1 fails for infinite powers. Let I be any set, and let $\mathcal{P}(I)$ denote the lattice of all subsets of I. The prime filters of $\mathcal{P}(I)$ are usually called *ultra filters*.

Example 2.3. For any set I and ultra filter $\mathcal{U} \subseteq \mathcal{P}(I)$, the map

$$f \colon \mathbb{C}^I \to \mathbb{C}, \ a \mapsto a_{\mathcal{U}} \text{ where } a^{-1}(a_{\mathcal{U}}) \in \mathcal{U}$$

is a homomorphism.

The following lemma gives that in fact, all non-constant homomorphisms from powers of $\mathbb{C}$ to $\mathbb{C}$ are of the form as in the preceding example, apart from an automorphism of $\mathbb{C}$.

Lemma 2.4. *Let I be an arbitrary set, and $f \colon \mathbb{C}^I \to \mathbb{C}$ a non-constant homomorphism. Then there exist an ultra filter $\mathcal{U} \subseteq \mathcal{P}(I)$ and an automorphism α of $\mathbb{C}$ such that for all $a \in C^I$*

$$f(a) = \alpha(a_{\mathcal{U}}) \text{ where } a^{-1}(a_{\mathcal{U}}) \in \mathcal{U}.$$

Proof. We assume first that $f(\chi_\emptyset) = 0$ and $f(\chi_I) = 1$. Note that in this case for all $X \subseteq I$ we have $\chi_\emptyset \to \chi_X \to \chi_I$, therefore $f(\chi_X) \in \{0, 1\}$. Let us define $\mathcal{U} = \{X \subseteq I : f(\chi_X) = 1\}$. We first show that $\mathcal{U}$ is an ultra filter. We prove this by showing that

(1) $\mathcal{U}$ is closed under finite intersections,
(2) for all $X \subseteq I$ we have either $X \in \mathcal{U}$ or $I \setminus X \in \mathcal{U}$,
(3) $\emptyset \notin \mathcal{U}$.

Item 3 is clear from the definition of $\mathcal{U}$. For the proofs of the other two items, let X and Y be any two subsets of I. We define an equivalence ϱ on I by

$$(i, j) \in \varrho \text{ iff } \chi_X(i) = \chi_X(j) \text{ and } \chi_Y(i) = \chi_Y(j).$$

Let k be the number of blocks of ϱ. Clearly, $k \leq 4$. Let $\mathbb{D}$ be the subdigraph spanned by

$$D = \{a \in C^I : (i, j) \in \rho \text{ implies } a(i) = a(j)\}$$

in $\mathbb{C}^I$. We require the following claim.

Claim. *The map $f|_D$ is a projection to some coordinate $s \in I$.*

Proof of claim. Let $\tau \colon \mathbb{C}^k \to \mathbb{D}$ be the isomorphism that maps $(c_1, \ldots, c_k)$ to $\sum_{i=1}^{k} c_i \chi_{B_i}$ where the B_i, $1 \leq i \leq k$, are the blocks of ϱ. Then by Corollary 2.2, $f|_D \tau \colon \mathbb{C}^k \to \mathbb{C}$ is a projection composed with an automorphism of $\mathbb{C}$. Since

$$\chi_\emptyset, \chi_I \in D, \ f(\chi_\emptyset) = 0 \text{ and } f(\chi_I) = 1,$$

we obtain that $f|_D \tau$ is a projection to some coordinate j where $1 \leq j \leq k$. Then $f|_D$ is a projection to any coordinate $s \in B_j$. ∎

To prove item 1, suppose that $X, Y \in \mathcal{U}$. Then by the claim $f|_D = \pi_s|_D$ for some coordinate $s \in I$. Now s must be in $X \cap Y$, since $\chi_X, \chi_Y \in D$, and

$$f(\chi_X) = \pi_s(\chi_X) = f(\chi_Y) = \pi_s(\chi_Y) = 1.$$

Thus $f(\chi_{X \cap Y}) = \pi_s(\chi_{X \cap Y}) = 1$.

To prove item 2, suppose that $X \notin \mathcal{U}$. By letting $X = Y$ and applying the claim, we obtain that there is an $s \in I$ such that $f|_D = \pi_s|_D$. This time s must be in $I \setminus X$, since $f(\chi_X) = \pi_s(\chi_X) = 0$. Hence $f(\chi_{I \setminus X}) = \pi_s(\chi_{I \setminus X}) = 1$. Thus $\mathcal{U}$ is an ultra filter indeed.

From the definition of $\mathcal{U}$, for any $a \in \{0, 1\}^I$, $f(a) = 1$ if and only if $a^{-1}(1) \in \mathcal{U}$. If $a \in \{0, 1\}^I$, we also have that $f(a) \in \{0, 1\}$, so $f(a) = 0$ if and only if $a^{-1}(0) \in \mathcal{U}$. Let $a \in C^I$ such that its range contains 2, say $a = \chi_A - \chi_B$ for two disjoint subsets A and B of I. Then the vertices $\chi_A + \chi_B$, a, χ_A clearly form a copy of $\mathbb{C}$ in $\mathbb{C}^I$. As $\chi_A + \chi_B, \chi_A \in \{0, 1\}^I$, $f(a)$ is uniquely determined by the known values $f(\chi_A + \chi_B)$ and $f(\chi_A)$. Thus f is the unique homomorphism from $\mathbb{C}^I$ to $\mathbb{C}$ extending the partial definition $f(\chi_\emptyset) = 0$ and $f(\chi_I) = 1$. Hence f must be the homomorphism given by $f(a) = a_\mathcal{U}$ for any $a \in C^I$.

For the general case we distinguish between two cases depending on whether $f(\chi_\emptyset) = f(\chi_I)$ or not. We first assume that $f(\chi_\emptyset) \neq f(\chi_I)$. As f is a homomorphism, $f(\chi_I) = f(\chi_\emptyset) + 1$. Let α be an automorphism of $\mathbb{C}$ such that $\alpha \colon f(\chi_\emptyset) \mapsto 0$. Then $\alpha^{-1} \circ f(\chi_\emptyset) = 0$ and $\alpha^{-1} \circ f(\chi_I) = 1$. Thus $\alpha^{-1} f(a) = a_\mathcal{U}$ for some ultra filter $\mathcal{U} \subseteq \mathcal{P}(I)$ which is exactly what we wanted to show.

We are left with the case when $f(\chi_\emptyset) = f(\chi_I)$. We claim that in this case f must be constant. Let a be an arbitrary element in C^I. Put $a = \chi_A - \chi_B$ for two disjoint subsets A and B of I. Let $b = \chi_A + \chi_B$ and $c = \chi_A$. Then

$$\chi_\emptyset \to b \to \chi_I, \chi_\emptyset \to c \to \chi_I \text{ and } b \to a \to c.$$

This is only possible if $f(a) = f(b) = f(c) = f(\chi_\emptyset) = f(\chi_I)$. □

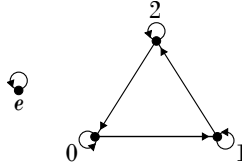The preceding lemma is needed in the present paper only to verify the following proposition.

FIGURE 3. The digraph $\mathbb{C}_1$.

Proposition 2.5. *Let $\mathbb{C}_1$ be the disjoint union of the one-element reflexive digraph and $\mathbb{C}$ where the isolated element of $\mathbb{C}_1$ is denoted by e, see Figure 3. Let $\mathbf{C}_1$ be*

an algebra on $C_1 = \{e, 0, 1, 2\}$ whose term operations coincide with the operations of $\mathrm{Pol}(\mathbb{C}_1)$. Then the variety $\mathcal{V}$ generated by $\mathbf{C}_1$ is a non-Taylor variety, and the non-trivial powers of $\mathbb{C}$ are not compatible digraphs in $\mathcal{V}$.

Proof. Since the Taylor identities (including idempotency) are preserved under taking retract and $\mathbb{C}$ that has no Taylor polymorphism is a retract of $\mathbb{C}_1$, $\mathcal{V}$ is a non-Taylor variety. To see that no non-trivial power of $\mathbb{C}$ is a compatible digraph in $\mathcal{V}$, suppose that $\mathbb{C}^I$ is compatible in $\mathcal{V}$ for some $I \neq \emptyset$. Observe that $\mathbb{C}_1$ admits a binary polymorphism f with a unit element e. Then $\mathbb{C}^I$ must admit a binary polymorphism f' with a unit element. We may conceive of f' as a homomorphism from $\mathbb{C}^{I \dot\cup I'}$ to $\mathbb{C}^I$ where I' is a disjoint copy of I. So $f' = (f'_i)_{i \in I}$ where for each $i \in I$, f'_i is a homomorphism from $\mathbb{C}^{I \dot\cup I'}$ to $\mathbb{C}$. Since f' has a unit element, for any $i \in I$, f'_i must depend on each of its coordinates in $I \dot\cup I'$. Let us fix an $i \in I$. Then by the preceding lemma, there exist an ultra filter $\mathcal{U} \subseteq \mathcal{P}(I \dot\cup I')$ and an automorphism α of $\mathbb{C}$ such that for all $a \in C^{I \dot\cup I'}$

$$f'_i(a) = \alpha(a_{\mathcal{U}}) \text{ where } a^{-1}(a_{\mathcal{U}}) \in \mathcal{U}.$$

Only one of I and I' is in $\mathcal{U}$, say $I \in \mathcal{U}$. Let $\bar{\mathcal{U}}$ be the restriction of $\mathcal{U}$ to I. Clearly, $\bar{\mathcal{U}}$ is an ultra filter on I. Moreover, for any $a \in \mathbb{C}^{I \dot\cup I'}$ if $b = a|_I$, then $b_{\bar{\mathcal{U}}} = a_{\mathcal{U}}$ and

$$f'_i(a) = \alpha(b_{\bar{\mathcal{U}}}) \text{ where } b^{-1}(b_{\bar{\mathcal{U}}}) \in \bar{\mathcal{U}}.$$

Thus, f'_i only depends on the coordinates in I, a contradiction. $\square$

The above proposition just shows that there exist non-Taylor varieties that have no non-trivial compatible $\mathbb{C}$-powers. So the non-Taylor varieties cannot be characterized by the existence of compatible $\mathbb{C}$-powers. In Section 4, we shall prove that such a characterization is possible by the use of a proper class of compatible disjoint unions of $\mathbb{C}$-powers. In the next section, we argue that there is no such characterization given by a single disjoint union of finite $\mathbb{C}$-powers. The proof of this fact uses a rather interesting Maltsev characterization of the n-th powers of relational structures.

3. A MALTSEV CONDITION CHARACTERIZING THE n-TH POWERS OF RELATIONAL STRUCTURES

In this section, relational structures are allowed to have infinitary relations. For any relational structure $\mathbb{A}$ and equivalence ν of A, we define the quotient relational structure $\mathbb{A}/\nu$ of the same signature as $\mathbb{A}$ as follows. The base set of $\mathbb{A}/\nu$ equals the set A/ν of the blocks of ν, and for any I-ary relation R of $\mathbb{A}$ the corresponding I-ary relation of $\mathbb{A}/\nu$ consists of the I-tuples $(a_i/\nu)_{i \in I}$ where $(a_i)_{i \in I} \in R$.

An operation $f \colon A^n \to A$ is called an *n-ary product decomposition operation* if f satisfies the following identities

$$f(x, \ldots, x) = x,$$
$$f(f(x_{1,1}, \ldots, x_{1,n}), \ldots, f(x_{n,1}, \ldots, x_{n,n})) = f(x_{1,1}, \ldots, x_{n,n}).$$

Remark 3.1. Let $\mathbb{A} = \prod_{j=1}^n \mathbb{B}_j$. Then the operation $f : A^n \to A$ defined by

$$f((b_{1,1}, \ldots, b_{1,n}), \ldots, (b_{n,1}, \ldots, b_{n,n})) = (b_{1,1}, \ldots, b_{n,n}) \text{ where } i, j \leq n, \ b_{i,j} \in B_j$$

is clearly an n-ary product decomposition polymorphism of $\mathbb{A}$.

Perhaps after this remark, it is not surprising that the existence of an n-ary product decomposition polymorphism leads to an n-fold product decomposition of a relational structure as follows.

Theorem 3.2. *Let $\mathbb{A}$ be any relational structure which has an n-ary product decomposition polymorphism f. For each $i \leq n$, let ν_i denote the binary relation that consists of the pairs $(a, b) \in A^2$ such that for some sequence c_j, $j \neq i$, in A*

$$f(c_1, \ldots, c_{i-1}, a, c_{i+1}, \ldots, c_n) = f(c_1, \ldots, c_{i-1}, b, c_{i+1}, \ldots, c_n).$$

Then the following hold.

(1) For each $i \leq n$, ν_i is an equivalence on A.
(2) The map

$$\varphi \colon A \to \prod_{i=1}^{n} A/\nu_i, \ a \mapsto (a/\nu_1, \ldots, a/\nu_n)$$

is an isomorphism from $\mathbb{A}$ to the relational structure $\prod_{i=1}^{n} \mathbb{A}/\nu_i$.

Proof. First we prove item (1). Let $i \leq n$. Clearly, ν_i is reflexive and symmetric. For proving transitivity, we require the following claim.

Claim. *For each $i \leq n$, ν_i consists of the pairs $(a, b) \in A^2$ such that for any sequence d_j, $j \neq i$, in A*

$$f(d_1, \ldots, d_{i-1}, a, d_{i+1}, \ldots, d_n) = f(d_1, \ldots, d_{i-1}, b, d_{i+1}, \ldots, d_n).$$

Proof of claim. Suppose that $a\nu_i b$. Then there exist c_j, $j \neq i$, such that

$$f(c_1, \ldots, c_{i-1}, a, c_{i+1}, \ldots, c_n) = f(c_1, \ldots, c_{i-1}, b, c_{i+1}, \ldots, c_n).$$

By using the fact that f is an n-ary product decomposition operation, we obtain that for any d_j, $j \leq n$,

$$f(f(d_1, d_2, \ldots, d_n), \ldots, f(c_1, \ldots, c_{i-1}, a, c_{i+1}, \ldots, c_n), \ldots, f(d_1, d_2, \ldots, d_n)) =$$
$$f(d_1, \ldots, d_{i-1}, a, d_{i+1}, \ldots, d_n)$$

and

$$f(f(d_1, d_2, \ldots, d_n), \ldots, f(c_1, \ldots, c_{i-1}, b, c_{i+1}, \ldots, c_n), \ldots, f(d_1, d_2, \ldots, d_n)) =$$
$$f(d_1, \ldots, d_{i-1}, b, d_{i+1}, \ldots, d_n).$$

Since the left sides of the preceding two equalities are equal, so are the right sides. This concludes the proof of the claim. ∎

Suppose now that $a\nu_i b\nu_i c$. Then by the claim, for any d_j, $j \neq i$,

$$f(d_1, \ldots, d_{i-1}, a, d_{i+1}, \ldots, d_n) = f(d_1, \ldots, d_{i-1}, b, d_{i+1}, \ldots, d_n) =$$
$$f(d_1, \ldots, d_{i-1}, c, d_{i+1}, \ldots, d_n).$$

In particular, $a\nu_i c$. Thus ν_i is transitive.

We prove item (2). Suppose that $\varphi(a) = \varphi(b)$. Then for all $i \leq n$, $a\nu_i b$. Since f is idempotent, by applying the above claim for the pair $(a, b) \in \nu_i$, $d_1 = \cdots = d_i = b$ and $d_{i+1} = \cdots = d_n = a$ where $i \leq n$ we obtain

$$a = f(a, a, \ldots, a) = f(b, a, \ldots, a) = f(b, b, a, \ldots, a) = \cdots = f(b, \ldots, b) = b.$$

So φ is injective.

To prove that φ is surjective, it suffices to show that for any sequence a_i, $i \leq n$, in A, $\varphi(f(a_1, \ldots, a_n)) = (a_1/\nu_1, \ldots, a_n/\nu_n)$, that is, for all i, $f(a_1, \ldots, a_n)\nu_i a_i$. We

prove the latter for $i = 1$, the proof is analogous for the cases when $i \geq 2$. By the properties of f

$$f(f(a_1, \ldots, a_n), a_2, \ldots, a_n) = f(f(a_1, \ldots, a_n), f(a_2, \ldots, a_2), \ldots, f(a_n, \ldots, a_n)) =$$
$$f(a_1, \ldots, a_n).$$

So $f(a_1, \ldots, a_n)\nu_1 a_1$ indeed.

Clearly, φ preserves the relations of $\mathbb{A}$. We prove that its inverse also does. Let ϱ be a J-ary relation of $\mathbb{A}$, and suppose that $(\varphi(a_j))_{j \in J} \in \varrho'$ where ϱ' is the relation of $\prod_{i=1}^{n} \mathbb{A}/\nu_i$ corresponding to ϱ. By the definition of ϱ', for each $i \leq n$ there is a tuple $(a_{j,i})_{j \in J} \in \varrho$ such that $a_j \nu_i a_{j,i}$ for all $j \in J$. Since f preserves ϱ, $(f(a_{j,1}, \ldots, a_{j,n}))_{j \in J} \in \varrho$. In the preceding paragraph we saw that $\varphi(f(a_{j,1}, \ldots, a_{j,n})) = (a_{j,1}/\nu_{j,1}, \ldots, a_{j,n}/\nu_{j,n})$. So by $a_j \nu_i a_{j,i}$, $i \leq n$, we have that $\varphi(a_j) = \varphi(f(a_{j,1}, \ldots, a_{j,n}))$. Since φ is injective, $a_j = f(a_{j,1}, \ldots, a_{j,n})$ for all $j \in J$. Thus $(a_j)_{j \in J} \in \varrho$, which concludes the proof. $\qquad \square$

Our main goal in this section is to obtain a characterization of the n-th powers of relational structures via some Maltsev condition. In order to do this, we require the following definition. An operation $f \colon A^n \to A$ is called an *n-ary power decomposition operation* if f is an n-ary product decomposition operation and there is a $g \colon A \to A$ such that f and g satisfy the additional identities

$$g^n(x) = x,$$
$$g(f(x_1, x_2, \ldots, x_n)) = f(g(x_2), \ldots, g(x_n), g(x_1)).$$

Such a g is called a *coordinate shift operation* with respect to f. In this case we also say that f is an n-ary power decomposition operation with respect to g.

Remark 3.3. Let $\mathbb{A} = \mathbb{B}^n$. Then the operation $f : A^n \to A$ defined by

$$f((b_{1,1}, \ldots, b_{1,n}), \ldots, (b_{n,1}, \ldots, b_{n,n})) = (b_{1,1}, \ldots, b_{n,n}) \text{ where } i, j \leq n, \ b_{i,j} \in B$$

is clearly an n-ary power decomposition polymorphism of $\mathbb{A}$ with the coordinate shift automorphism g defined by

$$g(b_1, b_2, \ldots, b_n) = (b_2, \ldots, b_n, b_1) \text{ where } i \leq n, \ b_i \in B.$$

Now we prove that the existence of an n-ary power decomposition polymorphism with repect to a coordinate shift endomorphism implies that the relational structure is isomorphic to the n-th power of some relational structure.

Theorem 3.4. *Let $\mathbb{A}$ be any relational structure which has an n-ary power decomposition polymorphism f with respect to a coordinate shift endomorphism g. For each $i \leq n$, let ν_i denote the equivalence defined in the statement of the preceding theorem. Then g is an automorphism of $\mathbb{A}$, and for any $i \leq n$, $g(\nu_i) = \nu_{i-1}$ where $i - 1$ is meant by modulo n. Moreover, $\mathbb{A}$ is isomorphic to the n-th power of the relational structure $\mathbb{A}/\nu_1$.*

Proof. Since g is an endomorphism with $g^n(x) = x$, g is an automorphism of $\mathbb{A}$ with inverse g^{n-1}. Let $i \leq n$. Let $(a, b) \in \nu_i$. we prove that $(g(a), g(b)) \in \nu_{i-1}$. By $(a, b) \in \nu_i$, there are some elements $c_i \in A, i \leq n$, such that

$$f(c_1, \ldots, c_{i-1}, a, c_{i+1}, \ldots, c_n) = f(c_1, \ldots, c_{i-1}, b, c_{i+1}, \ldots, c_n).$$

By plugging both sides into g and applying the identity satisfied by f and g we obtain

$$f(g(c_2), \ldots, g(c_{i-1}), g(a), g(c_{i+1}), \ldots, g(c_n), g(c_1)) =$$
$$f(g(c_2), \ldots, g(c_{i-1}), g(b), g(c_{i+1}), \ldots, g(c_n), g(c_1)).$$

Thus $(g(a), g(b)) \in \nu_{i-1}$, and hence $g(\nu_i) \subseteq \nu_{i-1}$ for all i. Then

$$\nu_{i-1} = g^n(\nu_{i-1}) \subseteq g^{n-1}(\nu_{i-2}) \subseteq \cdots \subseteq g^2(\nu_{i+1}) \subseteq g(\nu_i) \subseteq \nu_{i-1}$$

where the indices are meant by modulo n. Therefore, $g(\nu_i) = \nu_{i-1}$ for all $i \leq n$.

Finally, we prove that all of the factors $\mathbb{A}/\nu_i$, $i \leq n$, of the product decomposition of $\mathbb{A}$ are isomorphic to each other. Indeed, since g is an automorphism, for all $i \leq n$

$$\mathbb{A}/\nu_i \cong g(\mathbb{A})/g(\nu_i) \cong \mathbb{A}/\nu_{i-1}. \qquad \square$$

As an immediate consequence of the preceding remark and theorem, we obtain a characterization of the n-th powers by a strong Maltsev condition.

Theorem 3.5. *Any relational structure $\mathbb{A}$ is the n-th power of a relational structure if and only if $\mathbb{A}$ admits the strong Maltsev condition*

$$f(x, \ldots, x) = x,$$
$$f(f(x_{1,1}, \ldots, x_{1,n}), \ldots, f(x_{n,1}, \ldots, x_{n,n})) = f(x_{1,1}, \ldots, x_{n,n}),$$
$$g^n(x) = x,$$
$$g(f(x_1, x_2, \ldots, x_n)) = f(g(x_2), \ldots, g(x_n), g(x_1)).$$

We remark that for any n, the Maltsev condition that characterizes the n-fold product decomposition (the set of the first two identities for f in the corollary) is a trivial Maltsev condition. Indeed, by taking f as the first projection on the two element set, we obtain an n-ary product decomposition operation.

On the other hand, if a relational structure $\mathbb{A}$ of at least two elements admits the Maltsev condition for the n-th power decomposition (the set of all four identities in the corollary), then f must depend on each of its coordinates. This is because the equivalences ν_i, $i \leq n$, corresponding to f differ from the full relation A^2. In particular, for $n \geq 2$, the Maltsev condition that describes the n-th power decomposition is a non-trivial Maltsev condition. (This also follows from the fact that a 2-element set never is an n-th power for $n \geq 2$.)

Now we apply Theorem 3.5 to verify that certain digraphs are forbidden as compatible digraphs in the variety assigned naturally to the n-th power of a finite digraph of certain type.

Corollary 3.6. *Let $n \geq 2$, and let $\mathbb{A}$ be a finite, reflexive, connected, directly indecomposable digraph. Let $\mathbf{A}_n$ be the algebra on A^n with the polymorphism of $\mathbb{A}^n$ as basic operations, and $\mathcal{V}$ the variety generated by $\mathbf{A}_n$. Then $\mathcal{V}$ has no compatible digraph of the form $\bigcup_{j \in J} \mathbb{A}^{n_j}$ where $n_j < n$ for all $j \in J$ and $n_j > 0$ for some $j \in J$.*

Proof. Let us suppose to the contrary that $\mathcal{V}$ has a compatible digraph of the form $\bigcup_{j \in J} \mathbb{A}^{n_j}$. Since $\mathbb{A}^n$ and so $\mathcal{V}$ admit the Maltsev condition given in the preceding theorem, by the theorem there is a digraph $\mathbb{B}$ such that $\mathbb{B}^n \cong \bigcup_{j \in J} \mathbb{A}^{n_j}$. Now by the assumption for the disjoint union, at least one of the connected components of $\mathbb{B}$, say $\mathbb{B}_0$ has more than one element. Since the disjoint union is reflexive, $\mathbb{B}$ and

all of its components are reflexive. Hence $\mathbb{B}_0^n$ is a component of $\mathbb{B}^n$. So $\mathbb{B}_0^n \cong \mathbb{A}^{n_j}$ for some $n_j > 0$. Hence by the unique decomposition theorem for finite connected reflexive digraphs, see Corollary 4.7 in [8], $\mathbb{B}_0 \cong \mathbb{A}^m$ for some $m \geq 1$. This implies that $|A|^{mn} = |B_0|^n = |A|^{n_j}$ which contradicts the fact that $n > n_j$. $\square$

The immediate corollary below shows indeed that there exists no disjoint union $\mathbb{U}$ of finite $\mathbb{C}$-powers such that some of the $\mathbb{C}$-powers are non-trivial and $\mathbb{U}$ is a compatible digraph in any non-Taylor variety. In other words, the non-Taylor varieties cannot be characterized by the existence of a fixed compatible disjoint union of finite $\mathbb{C}$-powers.

Corollary 3.7. *Let $n \geq 2$. Let $\mathbf{C}_n$ be the algebra on C^n with the polymorphism of $\mathbb{C}^n$ as basic operations, and $\mathcal{V}$ the variety generated by $\mathbf{C}_n$. Then $\mathcal{V}$ has no compatible digraph of the form $\dot{\bigcup}_{j \in J} \mathbb{C}^{n_j}$ where $n_j < n$ for all $j \in J$ and $n_j > 0$ for some $j \in J$.*

4. Characterizations of Taylor varieties

In this section, we prove the main result of the paper. In order to do this, we require the following Lemma.

Lemma 4.1. *Let $\mathbb{G}$ be a relational structure and $\mathbb{K} \subseteq \mathbb{G}$. Let S be subclone of $\mathrm{Pol}(\mathbb{G})$. If for any $f \in \mathrm{Pol}(\mathbb{K})$, there is unique extension $f^* \in S$, then the map*

$$\varphi \colon \mathrm{Pol}(\mathbb{K}) \to S, \ f \mapsto f^*$$

is a clone homomorphism.

Proof. Clearly, any projection operation f on $\mathbb{K}$ extends to a projection operation of $\mathbb{G}$. By uniqueness of f^*, the latter projection operation must be f^*. We prove that φ commutes with composition. Let f be an n-ary and g be an m-ary operation in $\mathrm{Pol}(\mathbb{K})$. Then the restriction of $f^*(g^*(x_1, \ldots, x_m), y_2, \ldots, y_n)$ onto $\mathbb{H}$ is $f(g(x_1, \ldots, x_m), y_2, \ldots, y_n)$. So by uniqueness of the extension in S

$$(f(g(x_1, \ldots, x_m), y_2, \ldots, y_n))^* = f^*(g^*(x_1, \ldots, x_m), y_2, \ldots, y_n). \qquad \square$$

The following characterization of non-Taylor varieties plays a crucial role in the proof of our main result.

Lemma 4.2. *Let $\mathcal{V}$ be a variety. Then $\mathcal{V}$ is a non-Taylor variety if and only if there exist a non-empty set T and sets H_t, $t \in T$, such that some of the H_t are non-empty and $\dot{\bigcup}_{t \in T} \mathbb{C}^{H_t}$ is a compatible digraph in $\mathcal{V}$. If $\mathcal{V}$ is a locally finite non-Taylor variety, then T and H_t, $t \in T$, can be chosen to be finite.*

Proof. First we prove the "only if" part of the first statement of the lemma. So we assume that $\mathcal{V}$ is a non-Taylor variety. Let $\mathbf{F}$ be the free algebra freely generated by three elements x, y and z in $\mathcal{V}$. We define a compatible digraph $\mathbb{F}$ of $\mathbf{F}$ with letting the edge relation of $\mathbb{F}$ be the subalgebra of $\mathbf{F}^2$ generated by the set of pairs

$$\{(x, x), \ (y, y), \ (z, z), \ (x, y), \ (y, z), \ (z, x)\}.$$

By definition, for any $u, v \in F$, $u \to v$ in $\mathbb{F}$ if and only if there is a 6-ary term h of $\mathcal{V}$ such that $u(x, y, z) = h(x, y, z, x, y, z)$ and $v(x, y, z) = h(x, y, z, y, z, x)$ in $\mathbf{F}$.

Clearly, x, y and z are pairwise different elements in F. Indeed, being non-Taylor, $\mathcal{V}$ does not satisfy the identity $x = y$. Notice that the vertices x, y and z with the six edges of the above generating set constitute a subdigraph of $\mathbb{F}$ isomorphic to $\mathbb{C}$.

This subdigraph of $\mathbb{F}$ is called $\mathbb{C}_{\mathbb{F}}$. The subdigraph $\mathbb{C}_{\mathbb{F}}$ is a spanned subdigraph of $\mathbb{F}$, for otherwise, there would be a 6-ary term h such that

$$(h(x,y,z,x,y,z), h(x,y,z,y,z,x)) \in \{(y,x),(z,y),(x,z)\}$$

which is impossible, because the resulting identities, for example

$$h(x,y,z,x,y,z) = y \text{ and } h(x,y,z,y,z,x) = x,$$

make h to be a Taylor term for $\mathcal{V}$.

We conceive of the underlying set F of $\mathbf{F}$ as the set of ternary term operations of $\mathbf{F}$. Let T be the set of unary term operations of $\mathbf{F}$. For any $t \in T$, we write F_t for the set of the ternary term operations $u \in F$ such that $u(x,x,x) = t(x)$. Let $\mathbb{F}_t$ be the subdigraph of $\mathbb{F}$ spanned by F_t, $t \in T$. We claim that $\mathbb{F}_t$, $t \in T$, are exactly the connected components of $\mathbb{F}$. Indeed, if $u \to v$ in $\mathbb{F}$, then

$$u(x,x,x) = h(x,x,x,x,x,x) = v(x,x,x) = t(x)$$

for some 6-ary term h, and thus each connected component is contained in some F_t. On the other hand, by using the compatibility of $\mathbb{F}$ in $\mathbf{F}$, for any $t \in T$ and $u \in F_t$, we have $u(x,y,z) \to u(x,y,x)$ and $u(x,x,x) \to u(x,y,x)$. Hence for all $u \in F_t$, u and $t(x) = u(x,x,x)$ are in the same component of $\mathbb{F}$. Thus, for any $t \in T$, $\mathbb{F}_t$ is connected.

For any $t \in T$, let H_t denote the set of non-constant homomorphisms from $\mathbb{F}_t$ to $\mathbb{C}$. Let $T_0 := \{t \in T : H_t = \emptyset\}$ and $T^+ := T \setminus T_0$. For any $t \in T_0$, we define 0_t to be a new constant and for any $u \in F_t$, we let $[u] = 0_t$. For any $t \in T^+$ and for any $u \in F_t$, we let $[u]$ be the function $H_t \to \mathbb{C}, \varphi \mapsto \varphi(u)$.

For any $t \in T$, we define the digraph $\mathbb{K}_t$ as follows. The vertices of $\mathbb{K}_t$ are the $[u]$, $u \in F_t$, and the edges of $\mathbb{K}_t$ are the $([u],[v])$ where $u \to v$ in $\mathbb{F}_t$. We remark that by the definition, for any $t \in T$, $t \in T_0$ if and only if $\mathbb{K}_t$ is a one-element digraph, in which case the vertex set of $\mathbb{K}_t$ is $\{0_t\}$ and its edge relation is $\{(0_t, 0_t)\}$. We define $\mathbb{K}$ to be the digraph whose connected components coincide with the digraphs $\mathbb{K}_t$, $t \in T$, that is, $\mathbb{K}$ is the disjoint union of the $\mathbb{K}_t$, $t \in T$. Clearly, $\mathbb{K}$ is a reflexive subdigraph of the digraph $\mathbb{G} := \dot{\bigcup}_{t \in T} \mathbb{C}^{H_t}$ where the copy of $\mathbb{C}^{H_t}$ is identified with $\mathbb{K}_t$ whenever $t \in T_0$. We prove the required properties of the digraph $\mathbb{G}$ by the help of the next four claims.

Claim 1. *The kernel of the map $\psi \colon F \mapsto K, u \mapsto [u]$ is a congruence of $\mathbf{F}$.*

Proof of claim. To verify Claim 1, it suffices to show that $\ker(\psi)$ is preserved by every unary polynomial operation of $\mathbf{F}$. Every unary polynomial operation of $\mathbf{F}$ is of the form $s(x, u_2, \ldots, u_n)$ where $u_2, \ldots, u_n \in F$ and s is an n-ary term operation of $\mathbf{F}$. So let us assume that $(u,v) \in \ker(\psi)$. Then $u, v \in F_t$ for some t. We prove that $(s(u, u_2, \ldots, u_n), s(v, u_2, \ldots, u_n))$ is also in $\ker(\psi)$. As $\mathbb{F}$ is reflexive, the digraph $\hat{\mathbb{F}}_t = \mathbb{F}_t \times \{u_2\} \times \cdots \times \{u_n\}$ is isomorphic to $\mathbb{F}_t$, and hence $\hat{\mathbb{F}}_t$ is connected. Since $\hat{\mathbb{F}}_t \subseteq \mathbb{F}^n$ and s is a polymorphism of $\mathbb{F}$, $s(\hat{\mathbb{F}}_t)$ is also connected. Clearly, $s(u, u_2, \ldots, u_n), s(v, u_2, \ldots, u_n) \in s(\hat{F}_t)$. Hence, there exists $t' \in T$ such that $s(u, u_2, \ldots, u_n), s(v, u_2, \ldots, u_n) \in F_{t'}$. In the case when $t' \in T_0$, this yields immediately that $(s(u, u_2, \ldots, u_n), s(v, u_2, \ldots, u_n)) \in \ker(\psi)$. Now suppose that $t' \in T^+$, and let $\varphi \in H_{t'}$ be arbitrary. Then the map

$$\nu \colon F_t \mapsto \mathbb{C}, x \mapsto \varphi(s(x, u_2, \ldots, u_n))$$

is a homomorphism from $\mathbb{F}$ to $\mathbb{C}$. Since $(u, v) \in \ker(\psi)$, this implies in particular that $\nu(u) = \nu(v)$. Therefore $\varphi(s(u, u_2, \ldots, u_n)) = \varphi(s(v, u_2, \ldots, u_n))$. Since this holds for all $\varphi \in H_{t'}$, we obtain that $(s(u, u_2, \ldots, u_n), s(v, u_2, \ldots, u_n)) \in \ker(\psi)$. ∎

Now, we define an algebra $\mathbf{K}$ in $\mathcal{V}$ whose underlying set is K and whose basic operations are defined by

$$u_{\mathbf{K}}([x_1], \ldots, [x_n]) \coloneqq [u_{\mathbf{F}}(x_1, \ldots, x_n)]$$

for all basic operations $u_{\mathbf{F}}$ of $\mathbf{F}$ and $x_1, \ldots, x_n \in F$ provided $u_{\mathbf{F}}$ is n-ary. By Claim 1, the basic operations of $\mathbf{K}$ are well-defined. Now it is clear that ψ is an onto homomorphism from $\mathbf{F}$ to $\mathbf{K}$ and $\mathbf{K} \cong \mathbf{F}/\ker(\psi)$. Moreover, it is also clear from our construction that $\mathbb{K}$ is compatible with $\mathbf{K}$, and $\mathbb{K} \cong \mathbb{F}/\ker(\psi)$.

Claim 2. *The vertices $[x], [y], [z]$ are pairwise different.*

Proof of claim. As $\mathcal{V}$ is non-Taylor, its full idempotent reduct $\mathcal{V}_{\mathrm{Id}}$ interprets in the variety $\mathcal{SET}$. Let $\mathrm{id} \in T$ denote the identity term operation of $\mathbf{F}$. Recall that F_{id} is the set of all ternary idempotent term operations of $\mathbf{F}$. We define two algebras $\mathbf{A}$ and $\mathbf{B}$ on the same underlying set F_{id}. Algebra $\mathbf{A}$ is defined to be an algebra in $\mathcal{V}_{\mathrm{Id}}$ whose term operations are the restrictions of the idempotent term operations of $\mathbf{F}$ to F_{id}. Algebra $\mathbf{B}$ is defined to be the unique algebra on F_{id} in $\mathcal{SET}$. Notice that $\mathbf{A}$ is the free algebra freely generated by three elments in $\mathcal{V}_{\mathrm{Id}}$. Then, a set of identities in two variables that holds for the free algebra $\mathbf{A}$, also holds for $\mathcal{V}_{\mathrm{Id}}$. In particular, $\mathbf{A}$ has no Taylor term operation, since $\mathcal{V}_{\mathrm{Id}}$ interprets in $\mathcal{SET}$. So there is a clone homomorphism ζ from $\mathrm{Clo}(\mathbf{A})$ to $\mathrm{Clo}(\mathbf{B})$. We know that the vertices of $\mathbb{F}_{\mathrm{id}}$ are represented by the the idempotent ternary term operations of $\mathbf{F}$, and the edges of $\mathbb{F}_{\mathrm{id}}$ are given by certain identities of $\mathcal{V}_{\mathrm{Id}}$ of the form

$$u(x, y, z) = h(x, y, z, x, y, z) \text{ and } v(x, y, z) = h(x, y, z, y, z, x).$$

Now ζ preserves these identities, maps the ternary term operations of $\mathbf{A}$ to ternary projections and fixes the ternary projections on F_{id}. Hence ζ restricted to $F_{\mathrm{id}} \subseteq \mathrm{Clo}(\mathbf{A})$ is a retraction from $\mathbb{F}_{\mathrm{id}}$ onto $\mathbb{C}_{\mathbb{F}}$. This retraction composed with an isomorphism from $\mathbb{C}_{\mathbb{F}}$ to $\mathbb{C}$ yields a homomorphism from $\mathbb{F}_{\mathrm{id}}$ to $\mathbb{C}$. Now this homomorphism is in H_{id} and separates the elements x, y, z. Hence $[x], [y], [z] \in K$ are pairwise different as we claimed. ∎

Note that in $\mathbb{K}$ we have the edges $[x] \to [y]$, $[y] \to [z]$, $[z] \to [x]$, as well as all the loops. Also, since $\mathbb{K}_{\mathrm{id}} \subseteq \mathbb{C}^{H_{\mathrm{id}}}$, there are no more edges between the vertices $[x], [y], [z]$ in $\mathbb{K}$. This means in other words that the subdigraph induced on $\{[x], [y], [z]\}$ is isomorphic to $\mathbb{C}$. We write $\mathbb{C}_{\mathbb{K}}$ for this subdigraph.

Claim 3. *For any $t \in T$, $t \in T^+$ if and only if $|\{[t(x)], [t(y)], [t(z)]\}| = 3$.*

Proof of claim. The "if" direction of Claim 3 is obvious. Let us assume that $t \in T^+$, i.e. there exists a non-constant homomorphism φ from $\mathbb{F}_t$ to $\mathbb{C}$. Then $\varphi(u(x, y, z)) \neq \varphi(t(x))$ for some $u \in F_t$. So $[u(x, y, z)] \neq [t(x)] = [u(x, x, x)]$. Let u' denote the restriction of u to $\mathbb{C}_{\mathbb{K}}^3$. Since $\mathbb{K}$ is compatible in $\mathbf{K}$, u' is a homomorphism from $\mathbb{C}_{\mathbb{K}}^3$ to $\mathbb{K}_t \subseteq \mathbb{C}^{H_t}$, and u' is not constant since $[u(x, y, z)] = u([x], [y], [z])$ and $[u(x, x, x)] = u([x], [x], [x])$ are contained in the image $u(\mathbb{C}_{\mathbb{K}}^3)$ of u'. Then by Lemma 2.1, $u' = \iota \pi_J$ where π_J is the projection to the coordinates in J for some non-empty J, and ι is an isomorphism. Then π_J and hence u' map the diagonal elements of $\mathbb{C}_{\mathbb{K}}^3$ to separate elements. So $[t(x)] = u([x], [x], [x])$, $[t(y)] = u([y], [y], [y])$, and $[t(z)] = u([z], [z], [z])$ are pairwise different. ∎

Now let f be an n-ary operation in $\mathrm{Pol}(\mathbb{K})$. Clearly, f restricted to any connected component of $\mathbb{K}^n$ is a homomorphism to some component of $\mathbb{K}$. It is also clear that any component $\mathbb{D}$ of $\mathbb{K}^n$ equals some product $\prod_{j=1}^{n} \mathbb{K}_{t_j}$ where $t_1, \dots, t_n \in T$. So $f|_D$ maps $\prod_{j=1}^{n} \mathbb{K}_{t_j}$ to some $\mathbb{K}_t$, $t \in T$. Clearly, if $t \in T_0$, since $\mathbb{K}_t$ is one-element, $f|_D$ is a constant map. If $t \in T^+$, since $\mathbb{K}_t \subseteq \mathbb{C}^{H_t}$, $f|_D = (f_s)_{s \in H_t}$ where the f_s are homomorphisms from $\prod_{j=1}^{n} \mathbb{K}_{t_j}$ to $\mathbb{C}$.

Claim 4. *If $t \in T^+$, then any of the homomorphisms f_s as above is either constant or there exist a unique $1 \le \ell \le n$ with $t_\ell \in T^+$ and a unique homomorphism $\varphi \in H_{t_\ell}$ such that $f_s(y_1, \dots, y_n) = y_\ell(\varphi)$.*

Proof of claim. For the proof of Claim 4, let

$$\mathbb{P} := t_1(\mathbb{C}_{\mathbb{K}}) \times \cdots \times t_n(\mathbb{C}_{\mathbb{K}}).$$

Then we have $\mathbb{P} \subseteq \prod_{j=1}^{n} \mathbb{K}_{t_j}$. By Claim 3, we know that if $t_j \in T^+$ then $t_j(\mathbb{C}_{\mathbb{K}})$ is isomorphic to $\mathbb{C}$, otherwise $t_j(\mathbb{C}_{\mathbb{K}})$ is a singleton. Therefore, $\mathbb{P}$ is isomorphic to a power of $\mathbb{C}$. Hence, by Corollary 2.2, the restriction of f_s to P either is a constant map or there is a unique ℓ such that $t_\ell \in T^+$ and $f_s|_P$ only depends on the ℓ-th coordinate.

Let $([u_1], \dots, [u_n])$ be any tuple in $\prod_{j=1}^{n} \mathbb{K}_{t_j}$ where the u_j are arbitrary elements in F_{t_j} for $1 \le j \le n$. Note that in this case

$$\mathbb{P} \subseteq u_1(\mathbb{C}_{\mathbb{K}}^3) \times \cdots \times u_n(\mathbb{C}_{\mathbb{K}}^3) \subseteq \prod_{j=1}^{n} \mathbb{K}_{t_j}.$$

By Lemma 2.1, we know that each $u_1(\mathbb{C}_{\mathbb{K}}^3)$ is isomorphic to a finite power of $\mathbb{C}$, therefore

$$u_1(\mathbb{C}_{\mathbb{K}}^3) \times \cdots \times u_n(\mathbb{C}_{\mathbb{K}}^3)$$

is also isomorphic to a finite power of $\mathbb{C}$. Hence, by Corollary 2.2, the restriction of f_s to

$$u_1(C_{\mathbb{K}}^3) \times \cdots \times u_n(C_{\mathbb{K}}^3)$$

is a constant map or there is an $1 \le \ell' \le n$ such that $t_{\ell'} \in T^+$ and the restriction only depends on the ℓ'-th coordinate. Since $P \subseteq u_1(C_{\mathbb{K}}^3) \times \cdots \times u_n(C_{\mathbb{K}}^3)$, it follows that $\ell' = \ell$, that is, ℓ' does not depend on our choice of the tuple $([u_1], \dots, [u_n])$.

This implies, that either f_s is constant, or there exists a unique ℓ such that $t_\ell \in T^+$ and $f_s(y_1, \dots, y_n) = q(y_l)$ where q is a non-constant map from $\mathbb{K}_{t_\ell}$ to $\mathbb{C}$. In the latter case, q must be a homomorphism, since f_s is a homomorphism from the product of reflexive digraphs $\mathbb{K}_{t_j}$, $1 \le j \le n$. Let $\varphi := q \circ \psi$. Since $\psi(F_{t_\ell}) = K_{t_\ell}$, φ is not constant, and hence $\varphi \in H_{t_\ell}$. Then q must be the projection to the coordinate φ. Indeed, for any element $u \in F_{t_\ell}$

$$q([u]) = q(\psi(u)) = \varphi(u) = [u](\varphi).$$

The uniqueness of φ is now obvious. ∎

By Claim 2, $H_{\mathrm{id}} \ne \emptyset$. So to finish the proof of the "only if" part of the first statement of the lemma, we prove that $\mathbb{G} = \dot{\bigcup}_{t \in T} \mathbb{C}^{H_t}$ is a compatible digraph in $\mathcal{V}$.

As $\mathbb{K}$ is compatible in $\mathcal{V}$, it suffices to verify that there is a clone homomorphism from $\mathrm{Pol}(\mathbb{K})$ to $\mathrm{Pol}(\mathbb{G})$. We prove the existence of such a clone homomorphism by an application of Lemma 4.1 as follows. Clearly, $\mathbb{K} \subseteq \mathbb{G}$. We define a subclone S of $\mathrm{Pol}(\mathbb{G})$. The clone S consists of the n-ary operations $g \in \mathrm{Pol}(\mathbb{G})$) such that for each component $\mathbb{D}$ of the digraph $\mathbb{G}^n$ the restriction $g|_D$ is either constant or $g(D) \subseteq C^{H_t}$ for some $t \in T^+$ and each of the coordinate maps of $g|_D$ is a projection to some coordinate or constant. It is easy to see that S is indeed a subclone. By Claim 4, every n-ary operation of $f \in \mathrm{Pol}(\mathbb{K})$ uniquely extends to an operation of $f^* \in S$. So by Lemma 4.1, it follows that there exists a clone homomorphism from $\mathrm{Pol}(\mathbb{K})$ to S, and hence to $\mathrm{Pol}(\mathbb{G})$.

To conclude the proof of the first statement of the lemma, we assume that $\dot{\bigcup}_{t \in T} \mathbb{C}^{H_t}$ is a compatible digraph in $\mathcal{V}$ and for some $t \in T$, $H_t \neq \emptyset$. Then $\mathbb{C}$ is a retract of $\dot{\bigcup}_{t \in T} \mathbb{C}^{H_t}$. As the existence of a Taylor polymorphism is inherited under taking retract and $\mathbb{C}$ admits no Taylor operation, $\mathcal{V}$ is a non-Taylor variety.

The finiteness statement of the lemma is obvious by our construction of the $\mathbb{C}$-power components of $\mathbb{G}$ in the above proof. $\qquad\square$

In order to prove the primeness of the filter of Taylor interpretability types in $\mathcal{L}$, we require a modified version of the preceding lemma.

Corollary 4.3. *Let $\mathcal{V}$ be a variety. Then $\mathcal{V}$ is a non-Taylor variety if and only if for any large enough cardinal κ, the digraph $\dot{\bigcup}_{\lambda \leq \kappa} 2^\kappa \mathbb{C}^\lambda$, where λ stands for a cardinal and $2^\kappa \mathbb{C}^\lambda$ denotes the disjoint union of 2^κ-many copies of $\mathbb{C}^\lambda$, is a compatible digraph in $\mathcal{V}$.*

Proof. We prove the "only if" part of the claim, the other direction is immediate from the preceding lemma. Let $\mathcal{V}$ be an arbitrary non-Taylor variety. Since $\mathcal{V}$ is a non-Taylor variety, by the preceding lemma, there are some cardinals κ_t, $t \in T$, not all zero, such that $\mathbb{G} = \dot{\bigcup}_{t \in T} \mathbb{C}^{\kappa_t}$ is a compatible digraph in $\mathcal{V}$. Let κ be an arbitrary infinite cardinal with $\kappa \geq \kappa_t$ for all $t \in T$ and $\kappa \geq |T|$. By taking the κ-th power of $\mathbb{G}$, we obtain a compatible digraph $\mathbb{G}_1$ in $\mathcal{V}$ such that $\mathbb{G}_1$ is also a disjoint union of $\mathbb{C}$-powers and the largest exponent that appears in the $\mathbb{C}$-power components of $\mathbb{G}_1$ is κ. Clearly, $\mathbb{G}_1$ has at most 2^κ components and $|\mathbb{G}_1| = 2^\kappa$.

By a primitive positive definition as follows, we construct a compatible digraph $\mathbb{G}_2$ from $\mathbb{G}_1$ in $\mathcal{V}$ such that $\mathbb{G}_2$ is a sum of $\mathbb{C}$-powers where each cardinal $\lambda \leq \kappa$ appears as an exponent of some $\mathbb{C}$-power component. The vertex set of $\mathbb{G}_2$ is the set of homomorphisms from $\mathbb{C}$ to $\mathbb{G}_1$. To define the edge set of $\mathbb{G}_2$ we use the digraph $\mathbb{D}$ given in Figure 4. We set $f \to g$ in $\mathbb{G}_2$ iff the map given by $a_i \mapsto f(i)$, $b_i \mapsto g(i)$, $0 \leq i \leq 2$, is a homomorphism from $\mathbb{D}$ to $\mathbb{G}_1$.
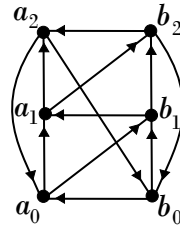

FIGURE 4. The digraph $\mathbb{D}$.

It is easy to see that $\mathbb{G}_2$ is a reflexive digraph. Since $\mathbb{D}$ is connected, any connected component of $\mathbb{G}_2$ is contained in the set of the homomorphisms from $\mathbb{C}$ to $\mathbb{C}^\nu$ where $\mathbb{C}^\nu$ is a connected component of the digraph $\mathbb{G}_1$. Let $(d_0, d_1, d_2) \in G_2$ an arbitrary element, that is, either $\{d_0, d_1, d_2\}$ is a singleton or induces a subdigraph isomorphic to $\mathbb{C}$ in $\mathbb{G}_1$. Our goal is to describe the connected component $\mathbb{E}$ of (d_0, d_1, d_2) in $\mathbb{G}_2$. In the argument what follows, we assume that all of the entries of $(d_0, d_1, d_2) \in G_2$ are in fact in C^ν.

Let s be any ordinal less than ν. Let us assume that $(d_0, d_1, d_2) \rightarrow (e_0, e_1, e_2)$ in $\mathbb{G}_2$. Then we have that $e_0, e_1, e_2 \in C^\nu$. Clearly, the map given by $a_i \mapsto d_i(s)$ and $b_i \mapsto e_i(s)$, $0 \le i \le 2$, is a homomorphism from $\mathbb{D}$ to $\mathbb{C}$. Then as one checks easily, for each $s < \nu$, either all of the $d_i(s)$ and $e_i(s)$ coincide where $0 \le i \le 2$, or the $d_i(s)$ are pairwise different. In the latter case, either $e_i(s) = d_i(s)$ for all $0 \le i \le 2$ or $e_i(s) = d_i(s) + 1$ for all $0 \le i \le 2$.

By the preceding paragraph, $\mathbb{E}$ consists of all 3-tuples $(e_0, e_1, e_2) \in G_2$ such that for each $s < \nu$, $e_0(s) = e_1(s) = e_2(s) = d_0(s)$ if $d_0(s) = d_1(s) = d_2(s)$ and $(e_0(s), e_1(s), e_2(s)) \in \{(0, 1, 2), (2, 0, 1), (1, 2, 0)\}$ otherwise. The edges in $\mathbb{E}$ are given by $(e_0, e_1, e_2) \rightarrow (f_0, f_1, f_2)$ iff for each $s < \nu$, either $e_i(s) = f_i(s)$ for all $0 \le i \le 2$ or $f_i(s) = e_i(s) + 1$ for all $0 \le i \le 2$. Then $\mathbb{E}$ is a product of one-element reflexive digraphs and copies of the subdigraph of $\mathbb{C}^3$ spanned by $\{(0, 1, 2), (2, 0, 1), (1, 2, 0)\}$ that is isomorphic to $\mathbb{C}$. Hence $\mathbb{E}$ is isomorphic to $\mathbb{C}^{\nu'}$ for some cardinal $\nu' \le \nu$. Clearly, for any cardinal $\nu' \le \nu$, if we set $(d_0(s), d_1(s), d_2(s)) = (0, 1, 2)$ for the coordinates $s < \nu'$ and $(d_0(s), d_1(s), d_2(s)) = (0, 0, 0)$ for the remaining coordinates $\nu' \le s < \nu$, then the component containing (d_0, d_1, d_2) in $\mathbb{G}_2$ is isomorphic to $\mathbb{C}^{\nu'}$. By taking into account that $\mathbb{G}_1$ has a component isomorphic to $\mathbb{C}^\kappa$, $\mathbb{G}_2$ has a component isomorphic to $\mathbb{C}^\lambda$ for any cardinal $\lambda \le \kappa$ and $\mathbb{G}_2$ is a disjoint union of such components. Notice that by our construction, any component of $\mathbb{G}_2$ has at most 2^κ isomorphic copies in $\mathbb{G}_2$. So $\mathbb{G}_2$ has at most $\kappa 2^\kappa = 2^\kappa$ components. Since each component of $\mathbb{G}_2$ has at most 2^κ elements, $\mathbb{G}_2$ is a compatible digraph of cardinality 2^κ in $\mathcal{V}$.

Let $\mathbb{L}$ denote the digraph with 2^κ-many vertices whose edge relation is the equality relation. Since $\mathbb{G}_2$ is a compatible digraph of cardinality 2^κ in $\mathcal{V}$, $\mathbb{L}$ is also a compatible digraph in $\mathcal{V}$. Finally, let $\mathbb{G}_3 = \mathbb{G}_2 \times \mathbb{L}$. Since both $\mathbb{G}_2$ and $\mathbb{L}$ are compatible in $\mathcal{V}$, so is $\mathbb{G}_3$. Clearly, $\mathbb{G}_3$ is a disjoint union of the powers $\mathbb{C}^\lambda$ when λ runs through the cardinals at most κ where each copy of $\mathbb{C}^\lambda$ appears precisely 2^κ-many times as a component of $\mathbb{G}_3$. $\square$

Now by the use of the preceding corollary, we can easily prove the main result of the paper.

Theorem 4.4. *The filter of the interpretability types of all Taylor varieties is prime in $\mathcal{L}$.*

Proof. It suffices to prove that the join of any two non-Taylor varieties is non-Taylor. Let $\mathcal{V}$ and $\mathcal{W}$ be arbitrary non-Taylor varieties. By the preceding corollary, there exists an infinite cardinal κ such that the digraph $\bigcup_{\lambda \le \kappa} 2^\kappa \mathbb{C}^\lambda$ is a compatible digraph in both varieties $\mathcal{V}$ and $\mathcal{W}$. Therefore, this digraph is also compatible in $\mathcal{V} \vee \mathcal{W}$. Then by using the corollary again, $\mathcal{V} \vee \mathcal{W}$ is a non-Taylor variety. $\square$

Finally, we would like to compare the interpretability types of the non-Taylor varieties that are defined via the digraphs occurring in the statement of Corollary 4.3. For an infinite cardinal κ, let $\mathcal{V}_\kappa$ denote the variety generated by the algebra $\mathbf{A}_\kappa$ whose universe is $A_\kappa = \dot{\bigcup}_{\lambda \leq \kappa} 2^\kappa C^\lambda$ where λ is a cardinal, and whose basic operations are the polymorphisms of $\mathbb{A}_\kappa = \dot{\bigcup}_{\lambda \leq \kappa} 2^\kappa \mathbb{C}^\lambda$. The *beth numbers* are the cardinals defined by the transfinite recursion $\beth_0 := \aleph_0$ and for any non-zero ordinal α, $\beth_\alpha := \sup_{\beta < \alpha} 2^{\beth_\beta}$ where β is an ordinal.

Proposition 4.5. *The following hold.*

(1) *For any two infinite cardinals κ_1 and κ_2 with $2^{\kappa_1} < 2^{\kappa_2}$, $\mathcal{V}_{\kappa_1}$ interprets in $\mathcal{V}_{\kappa_2}$, but $\mathcal{V}_{\kappa_2}$ does not interpret in $\mathcal{V}_{\kappa_1}$.*

(2) *The interpretability types of the non-Taylor varieties $\mathcal{V}_{\beth_\alpha}$ where α is an ordinal form a chain of unbounded size in $\mathcal{L}$.*

Proof. Item (1) implies item (2) obviously. So we prove that (1) holds. Within the proof, we use the abbreviations $\mathbb{A}_i = \mathbb{A}_{\kappa_i}$, $\mathbf{A}_i = \mathbf{A}_{\kappa_i}$, and $\mathcal{V}_i = \mathcal{V}_{\kappa_i}$ for $1 \leq i \leq 2$. Since

$$\mathbb{A}_1^{\kappa_2} = (\dot{\bigcup}_{\lambda \leq \kappa_1} 2^{\kappa_1} \mathbb{C}^\lambda)^{\kappa_2} \cong \dot{\bigcup}_{\lambda \leq \kappa_2} 2^{\kappa_2} \mathbb{C}^\lambda = \mathbb{A}_2,$$

$\mathbb{A}_2$ is a compatible digraph in $\mathcal{V}_1$. Therefore, $\mathcal{V}_1$ interprets in $\mathcal{V}_2$. We want to prove that $\mathcal{V}_2$ does not interpret in $\mathcal{V}_1$. To the contrary, let us assume that ζ is a clone homomorphism from the clone of $\mathcal{V}_2$ to the clone $\mathcal{V}_1$. Clearly, the unary constant operations in the clone of $\mathcal{V}_2$ are mapped to unary constant operations in the clone of $\mathcal{V}_1$ by ζ. Since $\mathbb{A}_2$ has 2^{κ_2} components, $\mathbb{A}_1$ has 2^{κ_1} elements, and $2^{\kappa_1} < 2^{\kappa_2}$, there exist two elements a and b in two different components of $\mathbb{A}_2$ such that $\zeta(\hat{a}) = \zeta(\hat{b})$ where $\hat{a}$ and $\hat{b}$ denote the unary constant term operations corresponding to a and b in the clone of $\mathcal{V}_2$, respectively. Clearly, the ternary operation $f(z, x, y)$ defined to be x if z is in the component containing a, and y otherwise is a basic operation of $\mathbf{A}_2$. Then the identities $f(\hat{a}, x, y) = x$, $f(\hat{b}, x, y) = y$ hold in the clone of $\mathcal{V}_2$. By applying ζ and using that $\zeta(\hat{a}) = \zeta(\hat{b})$, the identities

$$x = \zeta(f)(\zeta(\hat{a}), x, y) = \zeta(f)(\zeta(\hat{b}), x, y) = y$$

must be satisfied by the clone of $\mathcal{V}_1$. So in particular, $\mathbf{A}_1 \in \mathcal{V}_1$ must have one element, a contradiction. $\square$

In particular, it is not possible to characterize non-Taylor varieties by having a fixed compatible digraph of the form $\dot{\bigcup}_{t \in T} \mathbb{C}^{\kappa_t}$ where T is a non-empty set and the κ_t are cardinals, not all zero. Indeed, if $\dot{\bigcup}_{t \in T} \mathbb{C}^{\kappa_t}$ was compatible in all non-Taylor varieties, then by Corollary 4.3, there would be a κ such that $\mathbb{A}_\kappa$ would also be compatible, but by the first part of the preceding proposition $\mathbb{A}_\kappa$ is not compatible in the non-Taylor variety $\mathcal{V}_{2^\kappa}$, a contradiction. Notice that this behavior of non-Taylor varieties sharply differs from that of the idempotent non-Taylor varieties. Indeed, the latter varieties are characterized by having $\mathbb{C}$ as a compatible digraph.

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